SS 2016 Dr. Milena Pabiniak
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Differential Geometry

Homework 13

Mandatory Exercise 1. (10 points)
Show that any $2n$-dimensional complete Riemannian manifold (M, g) with constant positive sectional curvature $K = 1$ is isometric to (S^{2n}, g_{can}) or (RP^{2n}, g_{can}), where g_{can} denotes the canonical metric.
Hints: 1. Recall from the lecture that in this case the Riemannian universal cover of (M, g) is (S^{2n}, g_{can}).
2. Thus $M = S^{2n}/\Gamma$ where $\Gamma \subset Isom(S^{2n}, g_{can})$ is acting on S^{2n} in a totally discontinuous manner.
3. Analize eigenvalues and fixed points of elements of $Isom(S^{2n}, g_{can})$.

Mandatory Exercise 2. (10 points)
Let p be a prime number. Fix an integer k between 1 and $p-1$ and consider the following map on $S^3 = \{(z_1, z_2) \in \mathbb{C}^2;\ |z_1| = |z_2| = 1\}$ with the canonical metric,

$$t_{k,p}(z_1, z_2) = (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2k\pi i}{p}} z_2).$$

Show that $t_{k,p}$ gives a free and isometric action of $\mathbb{Z}_p = \{e^{\frac{2j\pi i}{p}};\ j = 0, 1, \ldots, p-1\}$. Let $L(p, k)$ denote the Riemannian manifold obtained from (S^3, g_{can}) by modding out by the above $\mathbb{Z}_p$ action. Show that $L(p, k)$ and $L(p, k')$ are isometric if and only if $k = \pm k'$ mod p or $k\,k' = \pm 1$ mod p.
Hint: Such manifolds are isometric if and only if the actions are conjugate by some isometry of S^3, i.e. if there exists an integer r such that $t_{k,p}$ and $t^r_{k',p}$ are conjugate. Analyze the eigenvalues.

Suggested Exercise 1. (0 points)
Take a Lie group G whose Lie algebra $\mathfrak{g}$ has a trivial center. Suppose G carries a bi-invariant Riemannian metric. Show that G and its universal cover are compact. Deduce that $\mathrm{SL}(n, \mathbb{R})$ carries no bi-invariant metric.
Hint: Recall from the lecture that if G has a bi-invariant metric then for any left-invariant vector fields X, Y, Z, W

$$R(X, Y, Z, W) = -\frac{1}{4} g([X, Y], [Z, W]).$$

Calculate the Ricci curvature $Ric(X, X)$ using an orthonormal basis $\{X_j\}$ of $T_e G = \mathfrak{g}$, and observe that the center being trivial implies that $Ric(X, X) > 0$ for all non-zero X.

Suggested Exercise 2. (0 points)
Let M^{2n} be an orientable, even dimensional Riemannian manifold with positive curvature. Let γ be a closed geodesic in M, i.e., an immersion of S^1 into M that is geodesic at all of its points. Prove that γ is homotopic to a closed curve whose length is strictly less than that of γ.
Hint: Look at the proof of theorems of Synge and Weinstein we discussed in class.

Suggested Exercise 3. (0 points)
A geodesic $\gamma\colon [0,\infty) \to M$ in a Riemannian manifold M is called a *ray starting from* $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s \in (0,\infty)$. Assume that M is complete and non-compact, and let $p \in M$. Show that M contains a ray starting from p.

Suggested Exercise 4. (0 points)
Let M be a complete Riemannian manifold, N a connected Riemannian manifold, and $f\colon M \to N$ a differentiable map which is locally an isometry. Assume that any two points of N can be joined by a unique geodesic in N. Prove that f is injective and surjective (and, therefore, f is a global isometry).

Suggested Exercise 5. (0 points)
Let $p \in M$ and let $f\colon M \to M$ be an isometry such that $f(p) = p$ and $df_p(v) = -v$ for all $v \in T_pM$. Let X be a parallel field along a geodesic γ in M with $\gamma(0) = p$. Show that $df_{\gamma(t)}X(\gamma(t)) = -X(\gamma(-t))$.
Hint: Note that $f(\gamma(t)) = \gamma(-t)$. Prove that $df_{\gamma(t)}X(\gamma(t))$ is parallel along $\gamma(t)$. Note that for $t = 0$ the desired equality holds, and use the uniqueness of parallel fields with given initial conditions.

Hand in: Monday 18th July
in the exercise session
in Seminar room 2, MI